Mathematics (80min.)

Ⅰ Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

Ⅱ Instructions for the Question Booklet

1. Do not open this question booklet until instructed to do so.
2. After being instructed, write your name and examination registration number in the space provided below, as shown on your admission ticket.

Ⅲ Instructions for the Answer Sheet

1. You must mark your answer on the answer sheet with an HB pencil.
2. Each answer column that is numbered [1] and so on in the questions represents a numeral from 0 to 9. [1] represents a single-digit number, [1][2] represents a two-digit number, and [1][2][3] represents a three-digit number.

Note the following:

(1) Express square roots ($\sqrt{\ }$) in their simplest form.
(Example: Substitute $4\sqrt{2}$ for $2\sqrt{8}$)

(2) For fractions, reduce the fraction to its lowest terms.

3. Carefully read the instructions on the answer sheet, too.

Examination registration number	
Name	

Mathematics (80min.)

Q1 Select one of the following options ① to ⑩ and fill the right number in the answer columns from [1] to [30]. The options can be used twice or more. [1]～[30]

(A)

Let $0° < \theta < 180°$.

(1) When $\tan\theta = -\frac{3}{4}$, $\sin\theta = \frac{\boxed{1}}{\boxed{2}}$.

(2) When $\sin\theta = \cos 50°$, $\theta =$ [3][4]°, [5][6][7]°.

(3) In isosceles triangle ABC, AB = AC, BC = 2, and angle BAC = 36°.

Point D is the intersection of the bisector of ∠ABC and side AC.

(i) ∠ABD = ∠CBD = [8][9]°.

(ii) AD = [10].

(iii) Since triangles ABC and BCD are similar, AB = [11] + $\sqrt{\boxed{12}}$.

(iv) $\sin 54° = \frac{\boxed{13} + \sqrt{\boxed{14}}}{\boxed{15}}$.

(B)

A die is rolled three times, and the numbers that appear are p, q, and r, in that order. Consider the parabola $C : y = px^2 + qx + r$.

(1) The probability that the y-intercept of C is 5 is $\dfrac{\boxed{16}}{\boxed{17}}$.

(2) The probability that C passes through the point $(1,\ 5)$ is $\dfrac{\boxed{18}}{\boxed{19}\boxed{20}}$.

(3) The probability that C is tangent to the x-axis is $\dfrac{\boxed{21}}{\boxed{22}\boxed{23}\boxed{24}}$.

(4) The probability that the x-coordinate of the vertex of C is an integer is $\dfrac{\boxed{25}}{\boxed{26}\boxed{27}}$.

(5) Given that the x-coordinate of the vertex of C is an integer, the conditional probability that the y-coordinate of the vertex is 0 is $\dfrac{\boxed{28}}{\boxed{29}\boxed{30}}$.

〔options in $\boxed{1}$～$\boxed{30}$〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q2 Select one of the following options ① to ⑩ and fill the right number in the answer columns from 31 to 58. The options can be used twice or more. 31 ~ 58

(A)

Let $f(x) = 8 \cdot 4^x + 2 \cdot 4^{-x} + 2^{x+1} + 2^{-x}$, and let $t = 2^{x+1} + 2^{-x}$.

(1) Let $2^x = u$. Then, $t = \boxed{31}u + \dfrac{\boxed{32}}{u}$, and t takes a minimum value of $\boxed{33}\sqrt{\boxed{34}}$.

At this point, $x = -\dfrac{\boxed{35}}{\boxed{36}}$. Also, when $t = 4$, $u = \boxed{37} \pm \dfrac{\sqrt{\boxed{38}}}{\boxed{39}}$

(2) $f(x) = \boxed{40}t^2 + t - \boxed{41}$.

(3) When the number of distinct real solutions for $f(x) = a$ is $1, a = \boxed{42} + \boxed{43}\sqrt{\boxed{44}}$.

(B)

On a plane, there is an equilateral triangle ABC with a side length of 9 and a point P. The point P satisfies the equation.

$$2\overrightarrow{AP} + 3\overrightarrow{BP} + 4\overrightarrow{CP} = \vec{0} \ .$$

Let G be the centroid of triangle ABC. Let D be the intersection of line AP and side BC, and let Q be the intersection of lines AG and CP.

(1) $\overrightarrow{AP} = \dfrac{\boxed{45}}{\boxed{46}}\overrightarrow{AB} + \dfrac{\boxed{47}}{\boxed{48}}\overrightarrow{AC}$.

(2) The ratio AP : PD = $\boxed{49}$: $\boxed{50}$, and BD : DC = $\boxed{51}$: $\boxed{52}$.

(3) GP = $\boxed{53}$.

(4) The ratio QP : PC = $\boxed{54}$: $\boxed{55}$.

(5) The area of triangle ABC is $\boxed{56}\boxed{57}\boxed{58}$ times the area of triangle GPQ.

〔options in $\boxed{31}$～$\boxed{58}$〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q3 Select one of the following options ① to ⑩ and fill the right number in the answer columns from 59 to 73. The options can be used twice or more. 59 ~ 73

(1) Let $f(x) = x^9 e^{-x}$, and let the maximum value of $f(x)$ be M.

$$f'(x) = \left(\boxed{59} x^{\boxed{60}} - x^{\boxed{61}} \right) e^{-x}.$$

The number of local extrema of $f(x)$ is 62. Furthermore, $f(x)$ reaches its maximum value M at $x = \boxed{63}$. Therefore, for $x > 0$,

$$0 < \frac{f(x)}{x} \leqq \frac{M}{x}$$

holds true. Hence, by the squeeze theorem,

$$\lim_{x \to \infty} x^8 e^{-x} = \boxed{64}.$$

(2) Let R be a positive constant, and define

$$I_1 = \int_0^R e^{-x}\, dx, \qquad I_s = \int_0^R x^{s-1} e^{-x}\, dx \quad (s > 1).$$

Also, define

$$G_s = \lim_{R \to \infty} I_s.$$

(i) We have

$$I_1 = \boxed{65} - e^{-R}, \quad G_1 = \boxed{66}.$$

For $s \geqq 2$, the following holds:

$$I_s = -R^{s-\boxed{67}} e^{-R} + \left(s - \boxed{68} \right) I_{s-1}$$

(ii) Finally,

$$G_4 = \boxed{69}, \quad G_8 = \boxed{70}\boxed{71}\boxed{72}\boxed{73}.$$

〔options in 59 ～ 73 〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q4 Select one of the following options ① to ⑩ and fill the right number in the answer columns from 74 to 83. The options can be used twice or more. 74 ~ 83

Let the parabola $C : y^2 = 12x$ have its directrix l and focus F.

(1) The equation of the directrix l is $x = -$ 74, and the coordinates of the focus F are $($ 75, 76 $)$.

(2) Let a be a positive constant. Consider a point $\mathrm{P}(a,\ 2\sqrt{3a})$ on the parabola C, and let m be the tangent line to C at P.
The equation of the tangent line m is:

$$\sqrt{a}\,y = \sqrt{\boxed{77}}\,(x + a).$$

Let Q be the point where m intersects the x-axis.
Then, $\mathrm{FP} = a +$ 78, $\mathrm{FQ} = a +$ 79.

Furthermore, let R be the point where m intersects the directrix l. The angle $\angle\mathrm{RFP} =$ 80 81 °.

(3) Let m_1 be a line passing through F and parallel to the y-axis. Let T_1 and T_2 be the points where m_1 intersects the parabola C.
The area of the triangle formed by m_1 and the two tangents to C at T_1 and T_2 is 82 83.

〔options in 74 ～ 83 〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

End of Mathematics questions. Leave the answer spaces 84 ～ 100 blank.

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