Mathematics (80min.)

Ⅰ Rules of Examination

1. Do not leave the room without proctor's permission.
2. Do not take this question booklet out of the room.

Ⅱ Instructions for the Question Booklet

1. Do not open this question booklet until instructed to do so.
2. After being instructed, write your name and examination registration number in the space provided below, as printed on your admission ticket.

Ⅲ Instructions for the Answer Sheet

1. You must mark your answer on the answer sheet with an HB pencil.
2. Each answer column that is numbered [1] and so on in the questions represents a numeral from 0 to 9. [1] represents a single-digit number, [1][2] represents a two-digit number, and [1][2][3] represents a three-digit number.

Note the following:

(1) Express square roots ($\sqrt{\ }$) in their simplest form.
(Example: Substitute $4\sqrt{2}$ for $2\sqrt{8}$)

(2) For fractions, reduce the fraction to its lowest terms.

3. Carefully read the instructions on the answer sheet, too.

Examination registration number	
Name	

Mathematics (80min.)

Q1 Select one of the following options ① to ⑩ and fill the right number in the answer columns from 1 to 24. The options can be used twice or more. 1 ~ 24

(A)

Let $f(x) = x^2 + 4x + 3$ and denote the parabola $y = f(x)$ as C. When the parabola C is translated by p units in the x-direction and q units in the y-direction, the resulting parabola C' is represented by $y = g(x)$. Here, p and q are real constants.

(1) When $p = 5$, $q = 2$, $g(x) = x^2 - \boxed{1}x + \boxed{2}\boxed{3}$.

(2) When the solution to the inequality $g(x) \leqq 0$ is $1 - \sqrt{2} \leqq x \leqq 1 + \sqrt{2}$, $p = \boxed{4}$, $q = -\boxed{5}$.

(3) When the minimum value of $g(x)$ is $\frac{3}{5}$, $q = \dfrac{\boxed{6}}{\boxed{7}}$.

(4) When the length of the line segment cut off by C' on the x-axis is $\sqrt{10}$, $q = -\dfrac{\boxed{8}}{\boxed{9}}$.

(5) Let $q = 0$. When the maximum and minimum values of $f(x)$ in $4 \leqq x \leqq 10$ and those of $g(x)$ in $3 \leqq x \leqq 9$ are the same, $p = -\boxed{10}$, $\boxed{11}\boxed{12}$.

(B)

(1) Among the positive integers less than or equal to 1000, the total number of multiples of 6 is 13 14 15.

(2) Among the positive integers less than or equal to 1000, the total number of numbers that are coprime with 6 is 16 17 18.

(3) Among the positive integers less than or equal to 1000, the total number of perfect squares is 19 20.

(4) Among the positive integers less than or equal to 100, the total number of numbers that are coprime with 80! is 21.

(5) Among the positive rational numbers less than or equal to 100, the total number of irreducible fractions with a denominator of 8 is 22 23 24.

〔options in 1 ～ 24〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q2 Select one of the following options ① to ⑩ and fill the right number in the answer columns from $\boxed{25}$ to $\boxed{52}$. The options can be used twice or more. $\boxed{25}$~$\boxed{52}$

(A)

Let $t = 2\sin\theta + \cos\theta$ and $f(\theta) = 4\sin 2\theta - 3\cos 2\theta - 6\sin\theta - 3\cos\theta$.

(1) Using an acute constant α, t can be expressed as $t = \sqrt{\boxed{25}}\sin(\theta + \alpha)$.

(2) $t^2 = \dfrac{\boxed{26}}{\boxed{27}} - \dfrac{\boxed{28}}{\boxed{29}}\cos 2\theta + \boxed{30}\sin 2\theta$.

(3) $f(\theta) = \boxed{31}t^2 - \boxed{32}t - \boxed{33}$.

(4) When $0 \leqq \theta \leqq \dfrac{\pi}{2}$, the maximum value of t is $\sqrt{\boxed{34}}$ and the minimum value is $\boxed{35}$.

(5) When $0 \leqq \theta \leqq \dfrac{\pi}{2}$, the minimum value of $f(\theta)$ is $-\boxed{36}$.

(B)

Let n be a positive integer, and let the region on the xy-plane defined by the system of inequalities $0 \leqq y \leqq f(x)$, $0 \leqq x \leqq n$ be D. Let S_n represent the number of lattice points in D. Here, a lattice point is defined as a point where both the x-coordinate and the y-coordinate are integers.

(1) $f(x) = x^2$. The number of lattice points in D where the x-coordinate is 0 is $\boxed{37}$, and the number of lattice points where the x-coordinate is 1 is $\boxed{38}$.

Therefore, $S_1 = \boxed{39}$.

(2) Let $f(x) = x^3$. Then, $S_9 = \boxed{40}\boxed{41}\boxed{42}\boxed{43}$.

(3) Let $f(x) = x^4 - (x-1)^4$. Then, $S_9 = \boxed{44}\boxed{45}\boxed{46}\boxed{47}$.

(4) Let $f(x) = \sqrt{x}$. Then, $S_{100} = \boxed{48}\boxed{49}\boxed{50}$.

(5) Let $f(x) = -x + 100$. The number of positive integers n not exceeding 100 for which S_n is a multiple of 3 is $\boxed{51}\boxed{52}$.

〔options in $\boxed{25}$～$\boxed{52}$〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q3 Select one of the following options ① to ⑩ and fill the right number in the answer columns from 53 to 68. The options can be used twice or more. 53 ~ 68

(1) Let $f(x) = x + \frac{4}{x^2}$.

The region D is the area enclosed by the curve $y = f(x)$,

the x-axis, and lines $x = 1$, and $x = 2$.

(i) The derivative of the function is given as $f'(x) = \boxed{53} - \boxed{54}x^{-\boxed{55}}$. Also, if $f'(x) = 0$, then $x = \boxed{56}$.

(ii) Furthermore, the following integrals hold:

$$\int_1^2 \frac{8}{x}\,dx = \boxed{57}\log\boxed{58}$$

$$\int_1^2 \frac{16}{x^4}\,dx = \frac{\boxed{59}\boxed{60}}{\boxed{61}}$$

Here, 58 is a prime number.

(iii) Given that D is rotated around the x-axis, the volume of the solid formed is $\left(\boxed{62}\log\boxed{63} + \boxed{64}\right)\pi$. Here, 63 is a prime number.

(2) Let $g(x) = x^3 - ax^2 + 4$, where a is a real constant.
The equation $g(x) = 0$ has three distinct real roots.
The range of a that satisfies this condition is $a > \boxed{65}$.

(3) Let $h(x) = x^4 - (a+1)x^3 + ax^2 + 4x - 4$.
Given that $h(1) = \boxed{66}$, the condition on the constant a such that the equation $h(x) = 0$ has three distinct real roots is $a = \boxed{67}$, and $a = \boxed{68}$, where $\boxed{67} < \boxed{68}$.

〔options in 53 ～ 68 〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

Q4 Select one of the following options ① to ⑩ and fill the right number in the answer columns from [69] to [79]. The options can be used twice or more. [69]~[79]

In the complex plane, three points $A(1+2i)$, $B(3+i)$, and $C(\alpha)$ are given, where i is the imaginary unit.
The argument θ is considered within the range $0 \leqq \theta < 2\pi$.

(1) When the magnitude of α is 4 and the argument is $\frac{\pi}{3}$, then $\alpha = [69] + [70]\sqrt{[71]}i$.
In this case, the magnitude of α^{10} is $2^{[72][73]}$, and the argument is $\frac{[74]}{[75]}\pi$.

(2) Given that triangle ABC is an isosceles right triangle with $\angle ABC = 90°$,
$\alpha = [76] + [77]i$, $[78] - i$.

(3) Consider the cubic equation $x^3 + px^2 + qx + r = 0$, where the three roots are $x = 1+2i$, $3+i$, and α. Here, p and q are complex constants, and r is a real constant.
If the argument of α is ϕ, then $\tan\phi = -$ [79].

〔options in 69 ～ 79 〕

① 1	② 2	③ 3	④ 4	⑤ 5
⑥ 6	⑦ 7	⑧ 8	⑨ 9	⑩ 0

End of Mathematics questions. Leave the answer spaces 80 ～ 100 blank.

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